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# Totally asymmetric attractive particle systems on $\mathbb{Z}$: hydrodynamic limit for general initial profiles

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**Abstract**

We extend previous results on the preservation of local equilibrium for one dimensional totally asymmetric attractive particle systems, the simple exclusion process and the zero range process. The hydrodynamic behavior is studied for general initial profiles when particles are jumping only to the nearest neighbor in a given direction.

*Key words:* Attractiveness; Coupling techniques; Law of large numbers; Local equilibrium; Hydrodynamic equation; Entropy solution; Euler scaling

## 1. Introduction

The study of the hydrodynamic behavior of interacting particle systems, which was introduced in (De Masi et al. 1984), is still incomplete for asymmetric processes. Here we deal with two attractive processes, the simple exclusion process (SEP) and the zero-range process (ZRP). For both, particles jump on the sites of $\mathbb{Z}$ following a translation invariant transition probability $p(x, y) = p(0, y - x)$ in such a way that particles do not jump on occupied sites for the first, and have a jumping rate function of the occupation number of their leaving site for the second. Assuming that the initial distribution is product and that $p(\cdot)$ has a non-zero first moment, the aim is to prove preservation of local equilibrium and to derive the hydrodynamic equation satisfied by the density profile. The first result of this type was obtained in (Rost, 1981) for the totally asymmetric SEP (i.e. $p(0, 1) = 1$) when the initial density profile is $\mathbb{1}_{(-\infty, 0)}$. The case of attractive systems with a one-step increasing or monotone decreasing initial

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profile was studied next [see (Liggett, 1985; Andjel and Kipnis, 1984; Andjel and Vares, 1987; Benassi and Fouque, 1987; 1988, 1992)].

This paper is a continuation of Benassi et al. (1991) where we considered monotone initial profiles for attractive systems. Our goal here is to apply the same reasoning to general initial profiles for totally asymmetric SEP and ZRP, that is to say first reduce the problem to step functions initial profiles (we do it in Section 3), then prove preservation of local equilibrium by induction on the number of steps and identify the limiting density profile as the unique entropy solution of the corresponding hydrodynamic equation.

However, while this work was in progress, Landim (1992) found a way to deduce preservation of local equilibrium from the previous paper (Rezakhanlou) (1991) where convergence of the density field for general bounded measurable initial profiles in $\mathbb{Z}^d$ was proved using entropy techniques. Landim used the regularity properties of the hydrodynamic equation, and thus covered a large class of asymmetric attractive processes in $\mathbb{Z}^d$, which includes totally asymmetric SEP and ZRP.

Considering this, we do not develop here the complete induction proof for preservation of local equilibrium but only its first step, i.e. the case of the simplest non-monotone initial profile, that we call a "bump" (in Section 4). Thus we present our method, which takes advantage of the fact that in the totally asymmetric case particles of different types do not get mixed up (and is based on couplings); this technique may be helpful to solve other problems.

## 2. Preliminaries and notations

For the simple exclusion process (SEP), the state space $E$ is the set of configurations $\{0, 1\}^{\mathbb{Z}}$ and the infinitesimal generator $L$ applied to a cylinder function $f$ (i.e. a function depending on finitely many coordinates) is given by

$$Lf(\eta) = \sum_{k \in \mathbb{Z}} \eta(k)[1 - \eta(k+1)][f(\eta^{k,k+1}) - f(\eta)], \tag{2.1}$$

where the configuration $\eta^{k,k+1}$ is obtained from configuration $\eta$ by exchanging occupations at sites $k$ and $k+1$.

For the zero range process (ZRP), the state space $E$ is the set of configurations $\mathbb{N}^{\mathbb{Z}}$ and the infinitesimal generator $L$ applied to a bounded cylinder function $f$ is given by

$$Lf(\eta) = \sum_{k \in \mathbb{Z}} \mathbb{1}_{\{\eta(k) > 0\}}[f(\eta^{k,k+1}) - f(\eta)], \tag{2.2}$$

where $\eta^{k,k+1}$ is obtained from configuration $\eta$ by taking a particle off site $k$ (if there is any) and adding it on site $k+1$.

In both cases, the associated semigroup is denoted by $(T_t)_{t \geq 0}$. For more details on these processes, we refer to Holley (1970), Andjel (1982), Liggett (1985).

For $k \in \mathbb{Z}$ we denote by $\tau_k$ the shift operator. It acts on $E$ by $\tau_k \eta(l) = \eta(l + k)$ (for $l \in \mathbb{Z}$), on functions by $\tau_k f(\eta) = f(\tau_k \eta)$ and on the set $\mathscr{P}$ of probability measures on $E$ by $(\tau_k \mu)(f) = \int f \mathrm{d}(\tau_k \mu) = \int (\tau_k f) \mathrm{d}\mu$ (for $f$ continuous and bounded). We recall that the set of extremal elements among the set of probability measures on $E$ which are shift invariant and invariant by the semigroup forms a one parameter family of product measures $(\nu^a)$ such that for $k \in \mathbb{Z}$, $n \in \mathbb{N}$

- for SEP: $0 \leq a \leq 1$ and $\nu^a\{\eta(k) = 1\} = a$,
- for ZRP: $0 \leq a < +\infty$ and $\nu^a\{\eta(k) = n\} = [a/(a+1)]^n[1/(a+1)]$.

We define $h(a)$ as the flow of particles through any site under the equilibrium measure $\nu^a$, i.e. for $k \in \mathbb{Z}$,

- for SEP: $h(a) = \int \eta(k)[1 - \eta(k+1)] \mathrm{d}\nu^a(\eta) = a(1-a)$,
- for ZRP: $h(a) = \int \mathbb{1}_{\{\eta(k) > 0\}} \mathrm{d}\nu^a(\eta) = a/(a+1)$.

Notice that $h$ is strictly concave. We recall the essential property of *attractiveness* (or *monotonicity*). The state space $E$ is endowed with the partial order $\eta \leq \xi$ if $\eta(k) \leq \xi(k)$ for every $k \in \mathbb{Z}$. It induces a stochastic order on $\mathscr{P}$ (see Liggett, 1985). Let $\mu_1, \mu_2$ be two elements of $\mathscr{P}$. Then $\mu_1 \leq \mu_2$ implies $T_t \mu_1 \leq T_t \mu_2$ for every $t \geq 0$: this is the *monotonicity property*, which will be our essential tool. We use it via *basic coupling* which consists in the construction on the same probability space of versions of the process starting from several arbitrary configurations, in such a way that particles of the coupled processes evolve together as much as possible (see Liggett (1985) for more details). We also use monotonicity via a different coupling, with *priorities*. This technique was introduced in Andjel and Kipnis (1984). In a coupled process $(\eta_t, \xi_t)$, we say that the $\eta$-particles have priority over the $\xi$-particles, and we note $\eta \vdash \xi$, when the process $(\eta_t, \eta_t + \xi_t)$ evolves according to the basic coupling; hence the $\eta$-particles evolve as if the $\xi$-particles were absent, the processes $(\eta_t)$ and $(\eta_t + \xi_t)$ evolve as the original process. More precisely, for the SEP and ZRP it means respectively that

- for SEP: If an $\eta$-particle wants to jump to a site occupied by a $\xi$-particle, they exchange their positions. But a $\xi$-particle cannot jump to a site occupied by an $\eta$-particle.
- for ZRP: If on a site $k$, $\eta$-particles and $\xi$-particles are present, the $\eta$-particles jump first.

We shall denote with an upper bar a coupled process, and mention explicitly the use of priorities (otherwise it will be basic coupling). Moreover we shall only deal with the $w^*$-convergence in $\mathscr{P}$: $\mu_n$ converges weakly to $\mu$ (when $n$ tends to $+\infty$) if for every bounded cylinder function $f$, $\int f \mathrm{d}\mu_n$ converges to $\int f \mathrm{d}\mu$. Finally, for $x \in \mathbb{R}$, $[x]$ denotes the integer part of $x$.

## 3. Main result and its reduction to step initial profiles

Let $u_0 : \mathbb{R} \to \mathbb{R}$ be a bounded (by 1 for the SEP) and locally of bounded variation function. For $\varepsilon > 0$ we define the *product distribution* $\mu_\varepsilon$ by its marginals

$$\mu_\varepsilon\{\eta(k) = n\} = \nu^{u_0(\varepsilon k)}\{\eta(0) = n\}, \tag{3.1}$$

with $k \in \mathbb{Z}$, $n \in \mathbb{N}$ for ZRP and $n \in \{0, 1\}$ for SEP. For every continuity point $x$ of $u_0$, $\tau_{[x\varepsilon^{-1}]}\mu_\varepsilon$ converges weakly to $\nu^{u_0(x)}$ as $\varepsilon$ tends to 0. We say that we have *local equilibrium* and that $\mu_\varepsilon$ has the initial density profile $u_0$. In the sequel we shall always use product distributions, given by their density profiles. Let $u(x, t)$ ($x \in \mathbb{R}$, $t \geq 0$) be the unique *entropy solution* of equation

$$\frac{\partial u}{\partial t} + \frac{\partial h(u)}{\partial x} = 0, \tag{3.2}$$

with initial condition

$$u(x, 0) = u_0(x),$$

i.e. the unique weak solution of this equation satisfying $u^-(x, t) \leq u^+(x, t)$ for every $x \in \mathbb{R}$, $t \geq 0$, where

$$u^-(x, t) = \lim_{y \uparrow x} u(y, t) \quad \text{and} \quad u^+(x, t) = \lim_{y \downarrow x} u(y, t).$$

It means that $u(x, t)$ has only increasing jumps in $x$ [for more details see (Benassi and Fouque, 1987; Andjel and Vares, 1987 and the references therein, for instance Lax, 1972)].

**Theorem 3.1.** *As $\varepsilon$ tends to 0, $\tau_{[x\varepsilon^{-1}]}T_{t\varepsilon^{-1}}\mu_\varepsilon$ converges weakly to $\nu^{u(x,t)}$ for every $t \geq 0$ and $x \in \mathbb{R}$ such that $u(\cdot, t)$ is continuous at $x$: local equilibrium is conserved.*

This theorem has been obtained in Andjel and Vares (1987) for $u_0$ continuous strictly decreasing and in Benassi et al. (1991) for $u_0$ monotone, under more general transition probabilities (without the nearest neighbor assumption). To prove Theorem 3.1, we first reduce it to the case where initial density profiles are step functions with compact support: we say that $u_0$ is an *n-step initial profile* ($n \geq 2$) if $u_0 = \sum_{i=1}^{n-1} a_i \mathbb{1}_{[x_{i-1}, x_i)}$ for a sequence $-\infty < x_0 < x_1 < \cdots < x_{n-1} < +\infty$ and densities $a_1, \ldots, a_{n-1}$ such that $a_i \neq a_{i+1}$ for every $i = 1, \ldots, n-2$, $a_1 \neq 0$ and $a_{n-1} \neq 0$.

**Lemma 3.2.** *Theorem* 3.1 *holds if for every n-step initial profile $v_0$ such that $\tilde{\mu}_\varepsilon$ has the density profile $v_0$,*

(a) *local equilibrium is conserved,*

(b) *at every discontinuity point x of the entropy solution* $v(\cdot, t)$ *of Eq.* (3.2) *with initial condition* $v_0$, *for every increasing cylinder function f*,

$$\int f \mathrm{d}\nu^{v^-(x,t)} \le \liminf_{\varepsilon \to 0} \int f \mathrm{d}(\tau_{[x\varepsilon^{-1}]} T_{t\varepsilon^{-1}} \tilde{\mu}_\varepsilon)$$

$$\le \limsup_{\varepsilon \to 0} \int f \mathrm{d}(\tau_{[x\varepsilon^{-1}]} T_{t\varepsilon^{-1}} \tilde{\mu}_\varepsilon) \le \int f \mathrm{d}\nu^{v^+(x,t)}. \tag{3.3}$$

**Proof.** We fix $t > 0$; we approximate the bounded and locally of bounded variation profile $u_0$ from below by increasing compact support profiles $u_n(\cdot, 0) = u_0(\cdot)\mathbb{1}_{[-n,n)}(\cdot)$, and we let $u(\cdot, t)$ (resp. $u_n(\cdot, t)$) be the entropy solution to Eq. (3.2) with initial condition $u_0$ (resp. $u_n(\cdot, 0)$). Then given a compact set $K \subset \mathbb{R}$, for $n$ large enough, $u_n(x, t) = u(x, t)$ if $x \in K$ (due to the hyperbolicity of Eq. (3.2) (see Krushkov, 1970, Section 3). We now fix $x \in \mathbb{R}$ and follow the proof of Andjel and Vares (1987, Lemma 3.3): we introduce a coupled process $(\eta_t, \xi_t, \zeta_t, \xi'_t, \zeta'_t)$ such that initially the $\xi$(resp. $\zeta$)-particles are on the right (resp. left) of the $\eta$-particles, the distribution $\mu_\varepsilon^n$ (resp. $\mu_\varepsilon$) of $\eta_0$ (resp. $\eta_o + \xi_0 + \zeta_0 = \eta_0 + \xi'_0 + \zeta'_0$) has density profile $u_n(\cdot, 0)$ (resp. $u_0$), $\xi_0 = \xi'_0$, $\zeta_0 = \zeta'_0$. We assume the priority $\eta \vdash \xi \vdash \zeta$; the $\xi'$ (resp. $\zeta'$)-particles follow independent continuous time rate 1 random walks with transition probability $p(0, -1) = 1$ (resp. $p(0, 1) = 1$). A law of large numbers for the leftmost (resp. rightmost) $\xi'$(resp. $\zeta'$)-particle gives, for any cylinder function $f$,

$$\lim_{\varepsilon \to 0} \int f(\xi + \zeta) \, \mathrm{d}(\tau_{[x\varepsilon^{-1}]} \bar{T}_{t\varepsilon^{-1}} \bar{\mu}_\varepsilon^n) = 0$$

provided that $n$ is large enough (so that $u_n(x, t) = u(x, t)$), where $\bar{\mu}_\varepsilon^n$ denotes the initial coupling distribution. It yields

$$\lim_{\varepsilon \to 0} \int f(\eta + \xi + \zeta) \mathrm{d}(\tau_{[x\varepsilon^{-1}]} \bar{T}_{t\varepsilon^{-1}} \bar{\mu}_\varepsilon^n) = \lim_{\varepsilon \to 0} \int f(\eta) \, \mathrm{d}(\tau_{[x\varepsilon^{-1}]} \bar{T}_{t\varepsilon^{-1}} \mu_\varepsilon^n)$$

$$= \int f \, \mathrm{d}\nu^{u_n(x,t)} = \int f \, \mathrm{d}\nu^{u(x,t)}$$

as soon as we prove conservation of local equilibrium for $\mu_\varepsilon^n$. (Notice that for the ZRP, we do not need to use $\xi$-particles, due to the definition of priority.)

We now therefore assume that $u_0$ has compact support. In [Lax, 1957, Section 2], $u(\cdot, t)$ is explicitly constructed at its continuity points, and the values $u^-(\cdot, t)$, $u^+(\cdot, t)$ can be easily deduced otherwise; moreover, if for a sequence of functions $(u_n(\cdot, 0), n \in \mathbb{N})$, $\int_0^y u_n(z, 0) \mathrm{d}z$ converges uniformly in $y \in \mathbb{R}$ to $\int_0^y u_0(z) \mathrm{d}z$ as $n$ goes to $+\infty$, and if for every $n \in \mathbb{N}$, $u_n(\cdot, t)$ is the entropy solution of (3.2) with initial condition $u_n(\cdot, 0)$, then at every continuity point $x$ of $u(\cdot, t)$, $u_n(x, t)$ (or $u_n^-(x, t)$ and $u_n^+(x, t)$ if $x$ is not a continuity point of $u_n(\cdot, t)$) converges to $u(x, t)$. So we approximate $u_0$ from below (resp. above) by a sequence $(\alpha_n(\cdot, 0), n \in \mathbb{N})$ (resp. $(\beta_n(\cdot, 0), n \in \mathbb{N})$) of step functions increasing (resp. decreasing) in $n$ for each $x \in \mathbb{R}$, and we let $\alpha_n(\cdot, t)$ (resp. $\beta_n(\cdot, t)$) be the entropy solution of Eq. (3.2) with initial condition $\alpha_n(\cdot, 0)$ (resp. $\beta_n(\cdot, 0)$). By monotonicity and the assumptions of the lemma on $(\alpha_n(\cdot, 0), n \in \mathbb{N})$ and $(\beta_n(\cdot, 0), n \in \mathbb{N})$, for an increasing cylinder function $f$, if $x$ is a continuity point of $u(\cdot, t)$, for $t \geq 0$, $n \in \mathbb{N}$ we have

$$\int f \, \mathrm{d}\nu^{\alpha_n^-(x,t)} \leq \liminf_{\varepsilon \to 0} \int f \, \mathrm{d}(\tau_{[x\varepsilon^{-1}]} T_{t\varepsilon^{-1}} \mu_\varepsilon)$$

$$\leq \limsup_{\varepsilon \to 0} \int f \, \mathrm{d}(\tau_{[x\varepsilon^{-1}]} T_{t\varepsilon^{-1}} \mu_\varepsilon) \leq \int f \, \mathrm{d}\nu^{\beta_n^+(x,t)}$$

Since

$$\lim_{n \to +\infty} \alpha_n^-(x, t) = u(x, t) = \lim_{n \to +\infty} \beta_n^+(x, t)$$

making $n$ tend to $+\infty$ gives conservation of local equilibrium from the initial density profile $u_0$. $\square$

Our method is then to prove, by induction on the number $n$ of steps, conservation of local equilibrium, property (3.3) and the following law of large numbers: for every $\delta > 0$, $x, y \in \mathbb{R}$ with $x < y$,

$$\lim_{\varepsilon \to 0} P_{\mu_\varepsilon} \left\{ \left| \int_x^y \tau_{[z\varepsilon^{-1}]} \eta_{t\varepsilon^{-1}}(0) \mathrm{d}z - \int_x^y u(z, t) \mathrm{d}z \right| > \delta \right\} = 0,$$

where $P_{\mu_\varepsilon}$ denotes the law of the process $(\eta_t)$ with initial distribution $\mu_\varepsilon$ (and $E_{\mu_\varepsilon}$ the corresponding expectation). But as mentioned in the introduction, we treat only the case of a "bump", namely a 2-step initial profile of the form $u_0 = a\mathbb{1}_{[-\alpha, 0)}$ with $\alpha > 0$ and $0 < a \leq 1$ for SEP or $a > 0$ for ZRP: this is the first step of our induction method. The next section is devoted to the proof of

**Proposition 3.3.** *Theorem* 3.1, *property* (3.3) *and the law of large numbers hold for "bumps".*

## 4. The case of a "bump"

We prove Proposition 3.3 in four steps. First we obtain the law of large numbers for a process denoted by $(\sigma_t)$, whose initial distribution is $\nu^{a,0}$, the product measure with density $v_0 = a\mathbb{1}_{(-\infty, 0)}$. In the second step, we couple the process $(\eta_t)$ under study with $(\sigma_t)$, we characterize an interface between the different types of particles, which converges in probability (thanks to the law of large numbers for $(\sigma_t)$, which is crucial in our proof). We can then deduce preservation of local equilibrium, (3.3), the fact that the limiting profile is the entropy solution of (3.2), and the law of large numbers for $(\eta_t)$.

### *4.1. Step 1. The law of large numbers for* $(\sigma_t)$

We know that Theorem 3.1 holds for the process $(\sigma_t)$. This corresponds to a decreasing Riemann problem for Eq. (3.2) and has been studied in Andjel and Vares (1987) and in [Benassi and Fouque, 1987, 1988 with the correction note (1992)]. Let $v_a(x,t)$ be the entropy solution of Eq. (3.2) starting from $v_0$.

**Proposition 4.1.** *For every* $x \in \mathbb{R}$, $t \geq 0$, $\delta > 0$,

$$\lim_{\varepsilon \to 0} P_{\nu^{a,0}}\left\{ \left| \int_x^{+\infty} \tau_{[y\varepsilon^{-1}]}\sigma_{t\varepsilon^{-1}}(0)\,\mathrm{d}y - \int_x^{+\infty} v_a(y,t)\,\mathrm{d}y \right| > \delta \right\} = 0. \tag{4.1}$$

**Proof.** We consider separately the SEP and the ZRP.

*Proof of* (4.1) *for the SEP.* In this case

$$v_a(y,t) = a\mathbb{1}_{(-\infty,(1-2a)t)}(y) + \tfrac{1}{2}\left(1 - \frac{y}{t}\right)\mathbb{1}_{[(1-2a)t,t)}(y).$$

For $a = 1$, using a subadditive ergodic theorem, Rost proved a strong version of (4.1) [see Rost, 1981 and also Liggett, 1985], i.e. that the convergence was almost sure and in $L^1(P_{\nu^{1,0}})$. Thus for $a < 1$, by monotonicity, for every $t \geq 0$ and $\delta > 0$, a first coupling of the processes with initial distributions $\nu^{a,0}$ and $\nu^{1,0}$ gives that

$$\lim_{\varepsilon \to 0} E_{\nu^{a,0}}\left\{ \int_t^{+\infty} \tau_{[y\varepsilon^{-1}]}\sigma_{t\varepsilon^{-1}}(0)\,\mathrm{d}y \right\} = 0, \tag{4.2a}$$

$$\lim_{\varepsilon \to 0} P_{\nu^{a,0}}\left\{ \int_x^{x'} \tau_{[y\varepsilon^{-1}]}\sigma_{t\varepsilon^{-1}}(0)\,\mathrm{d}y > \int_x^{x'} v_a(y,t)\mathrm{d}y + \delta \right\} = 0, \tag{4.2b}$$

for every $(1-2a)t \leq x < x'$, because $v_a(y,t) = v_1(y,t)$ for $y \geq (1-2a)t$, and a second coupling of the processes with initial distributions $\nu^{a,0}$ and $\nu^a$ gives that for every $x < x' \leq (1-2a)t$,

$$\lim_{\varepsilon \to 0} P_{\nu^{a,0}}\left\{ \int_x^{x'} \tau_{[y\varepsilon^{-1}]}\sigma_{t\varepsilon^{-1}}(0)\,\mathrm{d}y > \int_x^{x'} v_a(y,t)\mathrm{d}y + \delta \right\} = 0, \tag{4.2c}$$

since the strong law of large numbers holds for the initial equilibrium distribution $\nu^a$ (with the constant density profile $a$).

So, if (4.1) does not hold, for some $x_0 \le t_0$ and $\delta_0 > 0$, by (4.2a)–(4.2c),

$$\limsup_{\varepsilon \to 0} P_{\nu^{a,0}} \left\{ \int_{x_0}^{+\infty} \tau_{[y\varepsilon^{-1}]} \sigma_{t_0 \varepsilon^{-1}}(0)\, \mathrm{d}y < \int_{x_0}^{+\infty} v_a(y, t_0) \mathrm{d}y - \delta_0 \right\} \neq 0. \tag{4.3}$$

Let $x \le x_0$, $x < (1 - 2a)t_0$ and $x < 0$. By (4.2b), (4.2c),

$$\limsup_{\varepsilon \to 0} E_{\nu^{a,0}} \left\{ \int_{x}^{x_0} \tau_{[y\varepsilon^{-1}]} \sigma_{t_0 \varepsilon^{-1}}(0)\, \mathrm{d}y \right\} \le \int_{x}^{x_0} v_a(y, t_0) \mathrm{d}y,$$

and by (4.2a)–(4.2c), and (4.3),

$$\liminf_{\varepsilon \to 0} E_{\nu^{a,0}} \left\{ \int_{x_0}^{+\infty} \tau_{[y\varepsilon^{-1}]} \sigma_{t_0 \varepsilon^{-1}}(0)\, \mathrm{d}y \right\} = \liminf_{\varepsilon \to 0} E_{\nu^{a,0}} \left\{ \int_{x_0}^{t_0} \tau_{[y\varepsilon^{-1}]} \sigma_{t_0 \varepsilon^{-1}}(0)\, \mathrm{d}y \right\}$$

$$\le \int_{x_0}^{t_0} v_a(y, t_0) \mathrm{d}y - \delta_0 \limsup_{\varepsilon \to 0} P_{\nu^{a,0}} \left\{ \int_{x_0}^{+\infty} \tau_{[y\varepsilon^{-1}]} \sigma_{t_0 \varepsilon^{-1}}(0)\, \mathrm{d}y < \int_{x_0}^{+\infty} v_a(y, t_0) \mathrm{d}y - \delta_0 \right\}$$

$$< \int_{x_0}^{t_0} v_a(y, t_0) \mathrm{d}y,$$

therefore

$$\liminf_{\varepsilon \to 0} E_{\nu^{a,0}} \left\{ \int_{x}^{+\infty} \tau_{[y\varepsilon^{-1}]} \sigma_{t_0 \varepsilon^{-1}}(0)\, \mathrm{d}y \right\} < \int_{x}^{+\infty} v_a(y, t_0) \mathrm{d}y = -ax + a(1 - a)t_0. \tag{4.4}$$

But since we know that for $x < (1 - 2a)t_0$, $\tau_{[x\varepsilon^{-1}]} T_{t_0 \varepsilon^{-1}} \nu^{a,0}$ converges weakly to $\nu^a$ as $\varepsilon$ goes to 0 [see (Andjel and Vares, 1987) or (Benassi, Fouque, 1987, 1992)], we have

$$\lim_{\varepsilon \to 0} E_{\nu^{a,0}} \left\{ \int_{x}^{+\infty} \tau_{[y\varepsilon^{-1}]} \sigma_{t_0 \varepsilon^{-1}}(0)\, \mathrm{d}y \right\} = \lim_{\varepsilon \to 0} \left( E_{\nu^{a,0}} \left\{ \int_{x}^{+\infty} \tau_{[y\varepsilon^{-1}]} \sigma_0(0)\, \mathrm{d}y \right\} \right.$$

$$\left. + E_{\nu^{a,0}} \left\{ \int_0^{t_0} \sigma_{s\varepsilon^{-1}}([x\varepsilon^{-1}] - 1)(1 - \sigma_{s\varepsilon^{-1}}([x\varepsilon^{-1}]))\mathrm{d}s \right\} \right)$$

$$= -ax + a(1 - a)t_0$$

($x < 0$ and $x < (1 - 2a)t_0$ imply that $x < (1 - 2a)s$, for every $s \le t_0$), which contradicts (4.4); thus (4.1) is proved for the SEP.

*Proof of* (4.1) *for the ZRP.* In that case the unique entropy solution of Eq. (3.2) starting from $v_0$ is given by

$$v_a(y, t) = a\mathbb{1}_{(-\infty, t/(1+a)^2)}(y) + (\sqrt{t/y} - 1)\mathbb{1}_{[t/(1+a)^2, t)}(y).$$

Since the ZRP $(\sigma_t)$ is totally asymmetric, a coupling of the processes with initial distributions $\nu^{a,0}$ and $\nu^a$ gives that for every $t \geq 0$, $\delta > 0$ and $x < 0$,

$$\lim_{\varepsilon \to 0} P_{\nu^{a,0}} \left\{ \left| \int_x^0 \tau_{[y\varepsilon^{-1}]} \sigma_{t\varepsilon^{-1}}(0) \mathrm{d}y - ax \right| > \delta \right\} = 0. \tag{4.5.a}$$

The rightmost $\sigma$-particle behaves as a single particle performing a random walk totally asymmetric to the right thus

$$\lim_{\varepsilon \to 0} P_{\nu^{a,0}} \left\{ \int_t^{+\infty} \tau_{[y\varepsilon^{-1}]} \sigma_{t\varepsilon^{-1}}(0) \mathrm{d}y > \delta \right\} = 0. \tag{4.5.b}$$

To complete the proof of (4.1) we have to consider $\int_0^x \tau_{[y\varepsilon^{-1}]} \sigma_{t\varepsilon^{-1}}(0) \mathrm{d}y$ (with $x > 0$); for this we use the correspondence between SEP and ZRP, when particles only jump to their nearest neighbor site. First, because of the symmetry between particles and holes in the SEP we deduce from the preceding proof that (4.1) also holds for a SEP $(\zeta_t)$ of initial distribution $\nu^{b,1}$ which is totally asymmetric to the left, where $b = (1 + a)^{-1}$. For this SEP, the entropy solution of Eq. (3.2) starting from $b\mathbb{1}_{(-\infty,0)} + \mathbb{1}_{(0,+\infty)}$ is

$$w_b(y, t) = b\mathbb{1}_{(-\infty, -(1-2b)t)}(y) + \tfrac{1}{2}\left(1 + \frac{y}{t}\right)\mathbb{1}_{[-(1-2b)t, t)}(y) + \mathbb{1}_{(t, +\infty)}(y). \tag{4.6}$$

Because $\zeta$-particles only jump to their nearest neighbor site, their relative positions do not change. So we can number them: particle no. 0 is on site 0 at time 0 and we denote by $Z_t^{(0)}$ its position at time $t$, particle no. 1 is the first particle at the right of no. 0 ($Z_t^{(1)}$ is its position at time $t$), particle no. 2 is the second (of position $Z_t^{(2)}$), etc. Same way at the left of no. 0 we have no. $-1$, $-2$ (of positions $Z_t^{(-1)}$, $Z_t^{(-2)}$), etc. Then we can construct the processes $(\sigma_t)$ and $(\zeta_t)$ on the same probability space [see (Kipnis, 1986; Ferrari, 1986)] and write

$$\sigma_t(k) = Z_t^{(k+1)} - Z_t^{(k)} - 1 \quad \text{for } k \in \mathbb{Z},$$

$$\int_0^x \tau_{[y\varepsilon^{-1}]} \sigma_{t\varepsilon^{-1}}(0) \mathrm{d}y = \varepsilon\left(Z_{t\varepsilon^{-1}}^{([x\varepsilon^{-1}])} - Z_{t\varepsilon^{-1}}^{(0)} - [x\varepsilon^{-1}]\right)$$

$$+ (x - \varepsilon[x\varepsilon^{-1}])\left(Z_{t\varepsilon^{-1}}^{([x\varepsilon^{-1}]+1)} - Z_{t\varepsilon^{-1}}^{([x\varepsilon^{-1}])} - 1\right). \tag{4.7}$$

Hence we need to prove convergence for $\varepsilon Z_{t\varepsilon^{-1}}^{([x\varepsilon^{-1}])}$ and for $\varepsilon Z_{t\varepsilon^{-1}}^{(0)}$ (the last part of the right-hand side of (4.7) is less than $\varepsilon$). On the one hand, because the $(\zeta_t)$-process has initial distribution $\nu^{b,1}$ and is totally asymmetric to the left, particle n°0 behaves as a tagged particle under the equilibrium distribution $\nu^b$ therefore (see [Saada, 1987]): for every $\gamma > 0$

$$\lim_{\varepsilon \to 0} P_{\nu^{b,1}} \{ |\varepsilon Z_{t\varepsilon^{-1}}^{(0)} + (1 - b)t| > \gamma \} = 0. \tag{4.8}$$

On the other hand the profile $w_b(y,t)$ introduced in (4.6) is continuous and increases from the initial value $b>0$ to 1, so we can define $z(t)$ as the unique real number such that $\int_{-(1-b)t}^{z(t)} w_b(y,t)\mathrm{d}y = x$, and we now prove that: for every $\gamma > 0$

$$\lim_{\varepsilon\to 0} P_{\nu^{b,1}}\{|\varepsilon Z_{t\varepsilon^{-1}}^{([x\varepsilon^{-1}])} - z(t)| > \gamma\} = 0. \tag{4.9}$$

By (4.8) and because $\varepsilon[x\varepsilon^{-1}] = \int_{\varepsilon Z_{t\varepsilon^{-1}}^{(0)}}^{\varepsilon Z_{t\varepsilon^{-1}}^{([x\varepsilon^{-1}])}} \tau_{[y\varepsilon^{-1}]}\zeta_{t\varepsilon^{-1}}(0)\mathrm{d}y$,

$$\limsup_{\varepsilon\to 0} P_{\nu^{b,1}}\{\varepsilon Z_{t\varepsilon^{-1}}^{([x\varepsilon^{-1}])} > z(t) + \gamma\}$$

$$\le \limsup_{\varepsilon\to 0} P_{\nu^{b,1}}\{\varepsilon Z_{t\varepsilon^{-1}}^{([x\varepsilon^{-1}])} > z(t) + \gamma; |\varepsilon Z_{t\varepsilon^{-1}}^{(0)} + (1-b)t| < \gamma';$$

$$\varepsilon[x\varepsilon^{-1}] > \int_{-(1-b)t+\gamma'}^{z(t)+\gamma} \tau_{[y\varepsilon^{-1}]}\zeta_{t\varepsilon^{-1}}(0)\mathrm{d}y\}.$$

The last limit is equal to 0 by the law of large numbers for the process $(\zeta_t)$ and the definition of $z(t)$, if we choose $\gamma' < (b\gamma)/2$. Same way, $\limsup_{\varepsilon\to 0} P_{\nu^{b,1}}\{\varepsilon Z_{t\varepsilon^{-1}}^{([x\varepsilon^{-1}])} < z(t) - \gamma\} = 0$.

To conclude the proof of (4.1) for the ZRP, we compute the value of $z(t)$ according to its definition; then we can obtain via (4.7)–(4.9) the limit of $\int_0^x \tau_{[y\varepsilon^{-1}]}\sigma_{t\varepsilon^{-1}}(0)\mathrm{d}y$ and finally verify that this limit is equal to $\int_0^x v_a(y,t)\mathrm{d}y$,

- If $z(t)\in[-(1-b)t, -(1-2b)t)$, $z(t) = (x/b) - (1-b)t$, it corresponds to $x\in[0, b^2t)$ and $\int_0^x \tau_{[y\varepsilon^{-1}]}\sigma_{t\varepsilon^{-1}}(0)\mathrm{d}y$ converges to $ax = \int_0^x v_a(y,t)\mathrm{d}y$.
- If $z(t)\in[-(1-2b)t, t)$, $z(t) = 2\sqrt{tx} - t$, it corresponds to $x\in[b^2t, t)$ and $\int_0^x \tau_{[y\varepsilon^{-1}]}\sigma_{t\varepsilon^{-1}}(0)\mathrm{d}y$ converges to $-bt + 2\sqrt{tx} - x = \int_0^x v_a(y,t)\mathrm{d}y$.
- If $z(t)\ge t$, $z(t) = x$, thus $x\ge t$, and $\int_0^x \tau_{[y\varepsilon^{-1}]}\sigma_{t\varepsilon^{-1}}(0)\mathrm{d}y$ converges to $(1-b)t = \int_0^x v_a(y,t)\mathrm{d}y$.

### *4.2. Step 2. Definition and position of the interface*

We consider the coupled process $(\eta_t, \xi_t)$ with the priority $\eta \vdash \xi$. To obtain the initial coupling distribution $\bar{\mu}_\varepsilon$, at time $t=0$, $\xi$-particles are added to the left of $\eta$-particles in such a way that $\eta_0$ (resp. $\eta_0 + \xi_0$) has distribution $\mu_\varepsilon$ (resp. $\nu^{a,0}$). Thus $(\eta_t + \xi_t)$ is the process $(\sigma_t)$ introduced in Step 1, and (4.1) has the equivalent formulation: for every $x\in\mathbb{R}$, $t\ge 0$, $\delta > 0$,

$$\lim_{\varepsilon\to 0} P_{\bar{\mu}_\varepsilon}\left\{\left|\int_x^{+\infty} \tau_{[y\varepsilon^{-1}]}(\eta_{t\varepsilon^{-1}} + \xi_{t\varepsilon^{-1}})(0)\mathrm{d}y - \int_x^{+\infty} v_a(y,t)\mathrm{d}y\right| > \delta\right\} = 0. \tag{4.10}$$

Because $v_a(\cdot, t)$ is continuous and decreases from the initial value $a>0$ to 0, there is a unique real number $\alpha(t)$ such that $\int_{\alpha(t)}^{+\infty} v_a(y,t)\mathrm{d}y = \alpha a$.

We denote by $X_t^\varepsilon$ the position of the leftmost $\eta$-particle at time $t\varepsilon^{-1}$, thus

$$\begin{cases} \eta_{t\varepsilon^{-1}}(k) = 0 \text{ for } k < X_t^\varepsilon, \\ \xi_{t\varepsilon^{-1}}(k) = 0 \text{ for } k > X_t^\varepsilon, \end{cases} \tag{4.11}$$

since by our assumption on the dynamics particles jump only to their right nearest neighbor site. Notice that for ZRP it may happen that $\xi_{t\varepsilon^{-1}}(k) > 0$ on site $k = X_t^\varepsilon$. So $X_t^\varepsilon$ represents an interface between the different types of particles.

Furthermore the total number of particles is conserved by the dynamics

$$P_{\bar{\mu}_\varepsilon}\left\{\int_{-\infty}^{+\infty} \tau_{[y\varepsilon^{-1}]}\eta_0(0)\mathrm{d}y = \int_{-\infty}^{+\infty} \tau_{[y\varepsilon^{-1}]}\eta_{t\varepsilon^{-1}}(0)\mathrm{d}y\right\} = 1 \quad \text{for every } t \geq 0$$

therefore

$$\lim_{\varepsilon \to 0} P_{\bar{\mu}_\varepsilon}\left\{\left|\int_{-\infty}^{+\infty} \tau_{[y\varepsilon^{-1}]}\eta_0(0)\mathrm{d}y - \alpha a\right| > \delta'\right\} = 0 \quad \text{for every } \delta' > 0$$

and (4.11) imply that for every $t \geq 0$

$$\lim_{\varepsilon \to 0} P_{\bar{\mu}_\varepsilon}\left\{\left|\int_{\varepsilon X_t^\varepsilon}^{+\infty} \tau_{[y\varepsilon^{-1}]}\eta_{t\varepsilon^{-1}}(0)\mathrm{d}y - \alpha a\right| > \delta'\right\} = 0. \tag{4.12}$$

Then we have for every $\gamma > 0$

$$\lim_{\varepsilon \to 0} P_{\bar{\mu}_\varepsilon}\{|\varepsilon X_t^\varepsilon - \alpha(t)| > \gamma\} = 0. \tag{4.13}$$

Indeed

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} P_{\bar{\mu}_\varepsilon}\{\varepsilon X_t^\varepsilon > \alpha(t) + \gamma\} \\
&\quad = \limsup_{\varepsilon \to 0} P_{\bar{\mu}_\varepsilon}\left\{\varepsilon X_t^\varepsilon > \alpha(t) + \gamma;\ \alpha a - \delta' \leq \int_{\varepsilon X_t^\varepsilon}^{+\infty} \tau_{[y\varepsilon^{-1}]}\eta_{t\varepsilon^{-1}}(0)\mathrm{d}y\right\} \quad \text{by (4.12)} \\
&\quad = \limsup_{\varepsilon \to 0} P_{\bar{\mu}_\varepsilon}\left\{\varepsilon X_t^\varepsilon > \alpha(t) + \gamma;\right. \\
&\qquad \left.\alpha a - \delta' \leq \int_{\varepsilon X_t^\varepsilon}^{+\infty} \tau_{[y\varepsilon^{-1}]}(\eta_{t\varepsilon^{-1}} + \xi_{t\varepsilon^{-1}})(0)\mathrm{d}y\right\} \quad \text{by (4.11)} \\
&\quad \leq \limsup_{\varepsilon \to 0} P_{\bar{\mu}_\varepsilon}\left\{\varepsilon X_t^\varepsilon > \alpha(t) + \gamma; \alpha a - \delta' \leq \int_{\alpha(t)+\gamma}^{+\infty} \tau_{[y\varepsilon^{-1}]}(\eta_{t\varepsilon^{-1}} + \xi_{t\varepsilon^{-1}})(0)\mathrm{d}y\right\}
\end{aligned}$$

This limit is equal to 0 by (4.10) and the definition of $\alpha(t)$ if we choose $\delta'$ such that $\delta' < \int_{\alpha(t)}^{\alpha(t)+\gamma} v_a(y,t)\mathrm{d}y$ (it is possible since $v_a(\alpha(t),t) > 0$ and $v_a(\cdot,t)$ is continuous). Same way $\limsup_{\varepsilon \to 0} P_{\bar{\mu}_\varepsilon}\{\varepsilon X_t^\varepsilon < \alpha(t) - \gamma\} = 0$.

### 4.3. Step 3. The limiting profile

For every $x \in \mathbb{R}$, $t \geq 0$, we now define

$$u(x,t) = v_a(x,t)\mathbb{1}_{[\alpha(t),+\infty)}(x). \tag{4.14}$$

The point $x = \alpha(t)$ is clearly a discontinuity point of $u(\cdot,t)$ where by monotonicity and preservation of local equilibrium from the initial profile $v_0$, property (3.3) is satisfied. For every $x \neq \alpha(t)$, $\tau_{[x\varepsilon^{-1}]}T_{t\varepsilon^{-1}\mu_\varepsilon}$ converges weakly to $\nu^{u(x,t)}$ when $\varepsilon$ tends to 0 since by (4.13): for $x > \alpha(t)$

$$\lim_{\varepsilon\to 0}\tau_{[x\varepsilon^{-1}]}T_{t\varepsilon^{-1}}\mu_\varepsilon = \lim_{\varepsilon\to 0}\tau_{[x\varepsilon^{-1}]}T_{t\varepsilon^{-1}}\nu^{a,0} = \nu^{v_a(x,t)} = \nu^{u(x,t)},$$

and for $x < \alpha(t)$

$$\lim_{\varepsilon\to 0}\tau_{[x\varepsilon^{-1}]}T_{t\varepsilon^{-1}}\mu_\varepsilon = \nu^0.$$

This preservation of local equilibrium enables us to deduce that $u(x,t)$ is a weak solution of Eq. (3.2) with initial condition $u_0$. Let $\psi$ be a smooth function with compact support, and for $x \in \mathbb{R}$,

$$u_t^\varepsilon(x) = \int \eta(0)\mathrm{d}(\tau_{[x\varepsilon^{-1}]}T_{t\varepsilon^{-1}}\mu_\varepsilon).$$

Then

$$\frac{\partial u_t^\varepsilon}{\partial t}(x) = \varepsilon^{-1}\int L\eta(0)\mathrm{d}(\tau_{[x\varepsilon^{-1}]}T_{t\varepsilon^{-1}}\mu_\varepsilon) = \varepsilon^{-1}\int (g(\eta(-1)) - g(\eta(0))\mathrm{d}(\tau_{[x\varepsilon^{-1}]}T_{t\varepsilon^{-1}}\mu_\varepsilon)$$

with for $k \in \mathbb{Z}$

– for SEP: $g(\eta(k)) = \eta(k)(1 - \eta(k+1))$,

– for ZRP: $g(\eta(k)) = \mathbb{1}_{\{\eta(k)>0\}}$.

An integration by parts of $\int_{\mathbb{R}_+}\mathrm{d}t\int_{\mathbb{R}}\mathrm{d}x\,\psi(x,t)\dfrac{\partial u_t^\varepsilon}{\partial t}(x)$ and the change of variable $x = y + \varepsilon$ give

$$\int_{\mathbb{R}_+}\mathrm{d}t\int_{\mathbb{R}}\mathrm{d}x\left(\frac{\partial\psi}{\partial t}(x,t)u_t^\varepsilon(x) + \frac{1}{\varepsilon}(\psi(x+\varepsilon,t) - \psi(x,t))\int g(\eta(0))\,\mathrm{d}(\tau_{[x\varepsilon^{-1}]}T_{t\varepsilon^{-1}}\mu_\varepsilon)\right).$$

$$= -\int_{\mathbb{R}}\psi(x,0)u_0^\varepsilon(x)\mathrm{d}x.$$

Preservation of local equilibrium implies pointwise convergence at the continuity points of $u(\cdot, t)$ (when $\varepsilon$ goes to 0). Using that $\psi$ has compact support and the fact that $\int g(\eta(0))\mathrm{d}(\tau_{[x\varepsilon^{-1}]}T_{t\varepsilon^{-1}}\mu_\varepsilon)$ is bounded uniformly in $x$ and $t$, we conclude that $u(x,t)$ is a weak solution to Eq. (3.2). Next, $v_a(\cdot, t)$ being decreasing continuous, $u(\cdot, t)$ has only an increasing jump at $x = \alpha(t)$, therefore $u(x,t)$ is the unique entropy solution to Eq. (3.2) with initial condition $u_0$.

### *4.4. Step 4. The law of large numbers for* $(\eta_t)$

In order to prove

$$\lim_{\varepsilon\to 0} P_{\bar{\mu}_\varepsilon}\left\{\left|\int_x^{+\infty} \tau_{[y\varepsilon^{-1}]}\eta_{t\varepsilon^{-1}}(0)\mathrm{d}y - \int_x^{+\infty} u(y,t)\mathrm{d}y\right| > \delta\right\} = 0 \tag{4.15}$$

for every $x \in \mathbb{R}$, $t \geq 0$, $\delta > 0$, we consider three cases:

– For $x > \alpha(t)$, let $0 < \gamma < x - \alpha(t)$,

$$\limsup_{\varepsilon\to 0} P_{\bar{\mu}_\varepsilon}\left\{\left|\int_x^{+\infty} \tau_{[y\varepsilon^{-1}]}\eta_{t\varepsilon^{-1}}(0)\mathrm{d}y - \int_x^{+\infty} u(y,t)\mathrm{d}y\right| > \delta\right\}$$

$$= \limsup_{\varepsilon\to 0} P_{\bar{\mu}_\varepsilon}\left\{|\varepsilon X_t^\varepsilon - \alpha(t)| < \gamma;\right.$$
$$\left.\left|\int_x^{+\infty} \tau_{[y\varepsilon^{-1}]}\eta_{t\varepsilon^{-1}}(0)\mathrm{d}y - \int_x^{+\infty} u(y,t)\mathrm{d}y\right| > \delta\right\} \quad \text{by (4.13)}$$

$$= \limsup_{\varepsilon\to 0} P_{\bar{\mu}_\varepsilon}\left\{|\varepsilon X_t^\varepsilon - \alpha(t)| < \gamma;\right.$$
$$\left.\left|\int_x^{+\infty} \tau_{[y\varepsilon^{-1}]}(\eta_{t\varepsilon^{-1}} + \xi_{t\varepsilon^{-1}})(0)\mathrm{d}y - \int_x^{+\infty} v_a(y,t)\mathrm{d}y\right| > \delta\right\} \quad \text{by (4.11)}$$

and (4.14). This limit is equal to 0 by (4.10).

– For $x < \alpha(t)$, let $0 < \gamma < \alpha(t) - x$,

$$\limsup_{\varepsilon\to 0} P_{\bar{\mu}_\varepsilon}\left\{\left|\int_x^{+\infty} \tau_{[y\varepsilon^{-1}]}\eta_{t\varepsilon^{-1}}(0)\mathrm{d}y - \int_x^{+\infty} u(y,t)\mathrm{d}y\right| > \delta\right\}$$

$$\leq \limsup_{\varepsilon\to 0} P_{\bar{\mu}_\varepsilon}\left\{|\varepsilon X_t^\varepsilon - \alpha(t)| < \gamma; \int_x^{\varepsilon X_t^\varepsilon} \tau_{[y\varepsilon^{-1}]}\eta_{t\varepsilon^{-1}}(0)\mathrm{d}y > \delta/2\right\}$$

$$+ \limsup_{\varepsilon\to 0} P_{\bar{\mu}_\varepsilon}\left\{|\varepsilon X_t^\varepsilon - \alpha(t)| < \gamma; \left|\int_{\varepsilon X_t^\varepsilon}^{+\infty} \tau_{[y\varepsilon^{-1}]}\eta_{t\varepsilon^{-1}}(0)\mathrm{d}y\right.\right.$$

$$- \int_{\alpha(t)}^{+\infty} v_a(y,t)\mathrm{d}y \Big| > \delta/2 \Big\} \quad \text{by (4.14).}$$

These limits are equal to 0: the first one by (4.11), the second one by the definition of $\alpha(t)$ and (4.12).

– For $x = \alpha(t)$, let $0 < \gamma$,

$$\limsup_{\varepsilon \to 0} P_{\bar{\mu}_\varepsilon} \left\{ \left| \int_x^{+\infty} \tau_{[y\varepsilon^{-1}]} \eta_{t\varepsilon^{-1}}(0)\mathrm{d}y - \int_x^{+\infty} u(y,t)\mathrm{d}y \right| > \delta \right\}$$

$$\leq \limsup_{\varepsilon \to 0} P_{\bar{\mu}_\varepsilon} \left\{ \left| \int_{\alpha(t)-\gamma}^{\alpha(t)} \tau_{[y\varepsilon^{-1}]} \eta_{t\varepsilon^{-1}}(0)\mathrm{d}y - \int_{\alpha(t)-\gamma}^{\alpha(t)} v_a(y,t)\mathrm{d}y \right| > \delta/2 \right\}$$

$$+ \limsup_{\varepsilon \to 0} P_{\bar{\mu}_\varepsilon} \left\{ \int_{\alpha(t)-\gamma}^{+\infty} \tau_{[y\varepsilon^{-1}]} \eta_{t\varepsilon^{-1}}(0)\mathrm{d}y - \int_{\alpha(t)-\gamma}^{+\infty} u(y,t)\mathrm{d}y \right| > \delta/2 \right\}$$

$$\leq \limsup_{\varepsilon \to 0} P_{\bar{\mu}_\varepsilon} \left\{ \left| \int_{\alpha(t)-\gamma}^{\alpha(t)} \tau_{[y\varepsilon^{-1}]} (\eta_{t\varepsilon^{-1}} + \xi_{t\varepsilon^{-1}})(0)\mathrm{d}y > \delta/2 \right\}$$

$$+ \limsup_{\varepsilon \to 0} P_{\bar{\mu}_\varepsilon} \left\{ \left| \int_{\alpha(t)-\gamma}^{+\infty} \tau_{[y\varepsilon^{-1}]} \eta_{t\varepsilon^{-1}}(0)\mathrm{d}y - \int_{\alpha(t)-\gamma}^{+\infty} u(y,t)\mathrm{d}y \right| > \delta/2 \right\}$$

by monotonicity and the definition of $u(\cdot, t)$. These limits are equal to 0: in the first one we use (4.10) and $v_a(\cdot, t) \leq a$ (for every $t \geq 0$) to conclude by choosing $\gamma$ small enough; in the second one we use the previous result for $x < \alpha(t)$.

This completes the proof of proposition 3.3. □

The argument would be similar for the induction proof of Theorem 3.1. However it would involve several couplings and technical difficulties that we do not develop in this paper.

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